EUROPEAN BACCALAUREATE 2023

MARKING SCHEME

MATHEMATICS 3 PERIODS
PART A

DATE : 12th June 2023 Afternoon

DURATION OF THE EXAMINATION:

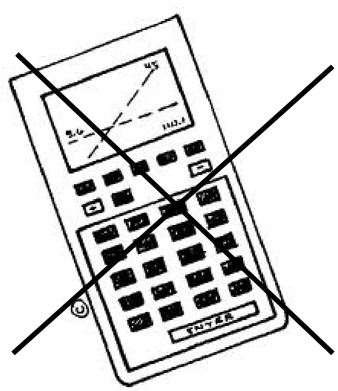
2 hours (120 minutes)

AUTHORIZED MATERIAL:

Examination without technological tool

Pencil for the graphs

Formelsammlung / Formula booklet / Recueil de formules

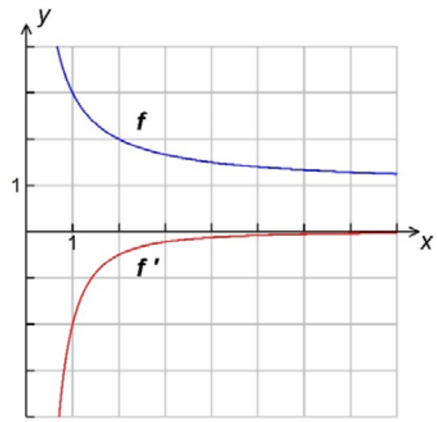


SPECIFIC INSTRUCTIONS:

- Answers must be supported by explanations.
- They must show the reasoning behind the results or solutions provided.
- If graphs are used to find a solution, they must be sketched as part of the answer.

- Unless indicated otherwise, full marks will not be awarded if a correct answer is not accompanied by supporting evidence or explanations of how the results or the solutions have been achieved.

- When the answer provided is not the correct one, some marks can be awarded if it is evident that an appropriate method and/or a correct approach has been used

PART A		
	Page 1/12	**Marks**
1) The diagram below shows the graph of a function *f* and its derivative f'.  **Determine** and **interpret** graphically: a) the average rate of change of the function *f* from $x_1 = 1$ to $x_2 = 2$.		2 marks
$P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points on the graph of *f*. From the graph: $y_1 = f(x_1) = 3$ and $y_2 = f(x_2) = 2$. The average rate of change of the function *f* from $x_1 = 1$ to $x_2 = 2$ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{2-1} = -1$. This result can be read directly on the graph: to get from P_1 to P_2, you can move one unit to the right (positive direction) parallel to the *x*-axis and then one unit down (negative direction) parallel to the *y*-axis. Thus $\frac{\Delta y}{\Delta x} = \frac{-1}{1} = -1$. It is the slope of the secant P_1P_2 to the graph of *f*.		
Calculating the average rate of change: 1 mark Interpreting graphically: 1 mark		

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b) the rate of change of the function *f* at $x_1 = 1$.		3 marks
The rate of change of the function *f* at $x_1 = 1$ equals $f'(1)$. From the graph: $f'(1) = -2$. It is the slope of the tangent to the graph of *f* at the point P_1 where $x_1 = 1$.		
Translating the rate of change at $x_1 = 1$ by $f'(1)$: 1 mark Reading $f'(1)$: 1 mark Interpreting graphically: 1 mark		

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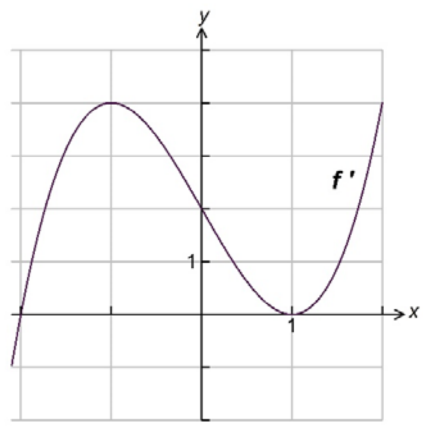
2) Consider a differentiable function *f*. The figure below shows the graph of its derivative f' for $-2.1 \le x \le 2$.  For each of the following statements **justify** whether it is true or false. a) The function *f* is decreasing for $-1 \le x \le 1$. b) The function *f* has a minimum at $x = -2$. c) There is a horizontal tangent to the graph of *f* at the point where $x = 1$. d) The slope of the tangent to the graph of *f* at the point where it intersects the *y*-axis is equal to 2. e) The graph of *f* has three horizontal tangents for $-2.1 \le x \le 2$.		5 marks
a) FALSE: For $-1 \le x \le 1$, $f'(x) \ge 0$. Therefore *f* is increasing for $-1 \le x \le 1$. b) TRUE: $f'(-2) = 0$ and f' changes sign when *x* passes through -2 (from $-$ to $+$). Hence *f* has a minimum at $x = -2$. c) TRUE: $f'(1) = 0$. There is therefore a horizontal tangent to the graph of *f* at the point where $x = 1$. d) TRUE: $f'(0) = 2$. Hence the slope of the tangent to the graph of *f* at the point where it intersects the *y*-axis is equal to 2. e) FALSE: $f'(x) = 0$ (when $-2.1 \le x \le 2$) $\Leftrightarrow x = -2$ or $x = 1$. Hence the graph of *f* has only two horizontal tangents for $-2.1 \le x \le 2$.		
1 mark for each statement		

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3) Consider the functions *f* and *F* defined by $f(x)=4x^3+3x^2$ and $F(x)=x^4+x^3+5$. a) **Show** that *F* is a primitive function of *f*.		2 marks
$F'(x)=(x^4+x^3+5)'=4x^3+3x^2=f(x).$ Hence *F* is a primitive function of *f.*		
b) **Calculate** $\int_1^2 f(x)dx$.		3 marks
F being a primitive of *f*, the function *G* defined by $G(x)=x^4+x^3$ is also a primitive function of *f.* Hence $\int_1^2 f(x)dx=\int_1^2 (4x^3+3x^2)dx=\left[x^4+x^3\right]_1^2=(16+8)-(1+1)=22.$ Note: Using *F* as a primitive of *f* when calculating the integral is also accepted.		
Writing correctly the integration: 2 marks Calculating the numerical value: 1 mark		

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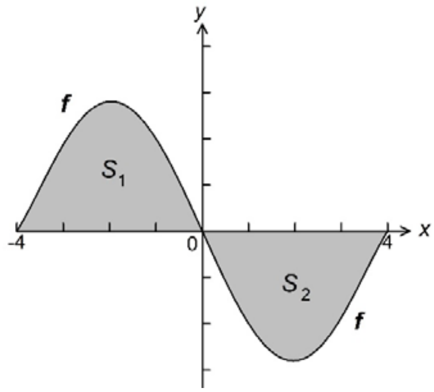
4) The figure below shows the graph of a function *f* and two regions S_1 and S_2 bounded by the graph of *f* and the *x*-axis.
The graph is symmetric with respect to the origin of the coordinate system.



You are given that $\int_{-4}^{0} f(x)dx = 7$.

a) **Interpret** the integral $\int_{-4}^{0} f(x)\,dx$ graphically. — 2 marks

$\int_{-4}^{0} f(x)dx$ is the area of the region bounded by the graph of *f* and the *x*-axes for $-4 \le x \le 0$, i.e. the area of S_1.

b) **Determine** — 3 marks

1. $\int_{0}^{4} f(x)dx$,
2. $\int_{-4}^{4} f(x)dx$,
3. the area of the region S_2.

1. $\int_{0}^{4} f(x)dx = -7$ (by symmetry of the graph with respect to the origin).
2. $\int_{-4}^{4} f(x)dx = \int_{-4}^{0} f(x)dx + \int_{0}^{4} f(x)dx = 7 + (-7) = 0$.
3. The regions S_1 and S_2 are symmetric with respect to the origin. S_2 has therefore the same area as S_1 i.e. 7 area units.

1 mark for each sub-question

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5) A swimming pool is being emptied and the volume of water that remains can be modelled by the function V given by $V(t) = 5000 \cdot 0.60^t \quad , \quad t \geq 0 \;,$ where time t is measured in hours and $V(t)$, measured in litres, is the volume of water, remaining at a time t. Emptying the pool starts at the time $t = 0$. a) **Determine** the volume of water in the pool at the start and after 1 hour.		2 marks
$V(0) = 5000 \cdot 0.60^0 = 5000.$ The volume of water in the pool at the start is 5000 litres. $V(1) = 5000 \cdot 0.60^1 = 3000.$ The volume of water in the pool after 1 hour is 3000 litres.		
1 mark for each volume		
b) **Calculate** the percentage rate at which the volume of water decreases per hour.		2 marks
$\frac{V(t+1)}{V(t)} = \frac{5000 \cdot 0.60^{t+1}}{5000 \cdot 0.60^t} = 0.60.$ (Note: This calculation is not required) In other words: in one hour the volume of water in the pool is multiplied by 0.60. The rate of decrease of the volume of water in the pool is therefore 40% per hour. Note: Or use the rule $a = 1 + r$, where a is the base and r the rate of change.		
Explaining: 1 mark Determining the required percentage: 1 mark		
c) **Explain** what the model tells us about the volume of water remaining after a very long time.		1 mark
$\lim_{t \to \infty} V(t) = 5000 \cdot 0 = 0.$ Therefore, according to the model, there will be no water remaining in the pool after an infinite time. Notes: Other answers must be accepted. For example: there will always be a small amount of water left because the zero limit does not mean that the 0-value is reached. It only tends towards zero. Candidates may also reflect on whether a model is realistic over an infinite time. Accept such answers even if it is not required in this question.		

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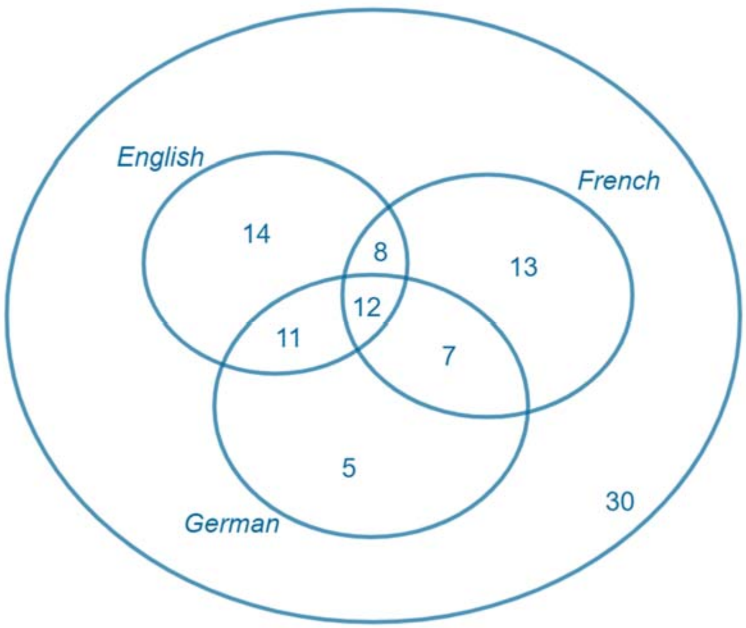
6) a) **Calculate** in how many ways the letters of the word PARIS can be ordered.		2 marks
The number of permutations of n distinct objects without repetition is $n!$ Thus the number of permutations of the 5 letters of the word PARIS is $5! = 120$. The 5 letters of the word PARIS can be ordered in 120 ways.		
Writing the right formula: 1 mark Calculating: 1 mark		
b) **Calculate** how many "words" (not necessarily having a meaning) of 3 different letters you can write using letters of the word PARIS.		3 marks
The number of permutations of k objects from a set of n distinct objects without repetition is $\frac{n!}{(n-k)!}$. Thus the number of permutations of 3 different letters from the 5 letters of PARIS is $\frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60$. We can write 60 "words" of 3 different letters chosen from the 5 letters of the word PARIS.		
Writing the right formula: 1 mark Calculating: 2 marks		

PART A		
	Page 8/12	**Marks**
7) A survey of 100 students enrolling at a university, shows that • 45 speak English • 40 speak French • 35 speak German • 20 speak both English and French • 23 speak both English and German • 19 speak both French and German • 12 speak all three languages. Using a Venn diagram or otherwise, **determine** the probability that a randomly selected student from these 100 students speaks only one of these three languages.		5 marks



$P(\text{only English or only French or only German}) =$
$P(\text{only English}) + P(\text{only French}) + P(\text{only German}) =$

$$\frac{14}{100} + \frac{13}{100} + \frac{5}{100} = \frac{32}{100}.$$

The probability that a randomly selected student from the 100 students speaks only one of the three languages equals $\frac{32}{100} = 0.32.$

Using a correct Venn diagram (or other method): 3 marks
Calculating the required probability: 2 marks

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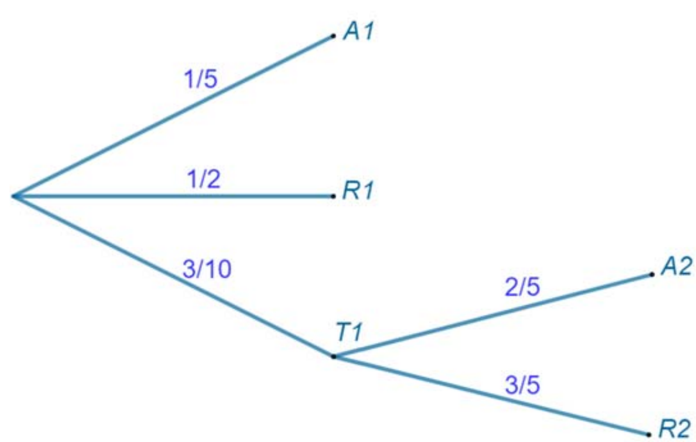
8) Applicants for jobs in a large company must sit an aptitude test. They are either

- accepted with a probability of $\frac{1}{5}$ or
- rejected with a probability of $\frac{1}{2}$ or
- retested with a probability of $\frac{3}{10}$.

When they are retested, there are just two outcomes, accepted with a probability of $\frac{2}{5}$ or rejected with a probability of $\frac{3}{5}$.

a) **Draw** a tree diagram to illustrate the outcomes. — 2 marks

Let the events be:
A_1: "accepted after the first try"
R_1: "rejected after the first try"
T_1: "retested after the first try"
A_2: "accepted after the second try"
R_2: "rejected after the second try"



b) **Determine** the probability that a randomly selected applicant is accepted. — 3 marks

$$P(\text{accepted}) = P(A_1) + P(T_1)\cdot P(A_2|T_1) = \frac{1}{5} + \frac{3}{10}\cdot\frac{2}{5} = \frac{5+3}{25} = \frac{8}{25} = 0.32.$$

The probability that a randomly selected candidate is accepted equals $\frac{8}{25}$ or 0.32.

Note: Candidates are free to use the diagram or the formulae.

Using correctly the diagram or the formulae: 2 marks
Calculating: 1 mark

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9) A biased coin is thrown several times. At each throw, the probability of getting a head is $\frac{1}{3}$. a) Is this a Bernoulli process? **Justify** your answer.		2 marks
Yes. We have a sequence of Bernoulli independent trials: the throws of the coin. Each throw of the coin is a Bernoulli trial, i.e. an experiment which has exactly two outcomes: success=head or failure=tail. The probability of success is the same at each trial: $p = \frac{1}{3}$.		
Yes: 0.5 mark 0.5 mark for each of the three elements of the justification		
b) The coin is thrown 3 times. **Calculate** the probability of getting exactly 2 heads.		2 marks
X = the number of heads. X is binomially distributed with parameters $n = 3$ and $p = \frac{1}{3}$. $P(X = 2) = \binom{3}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} = 3 \cdot \frac{1}{9} \cdot \frac{2}{3} = \frac{2}{9}$. The probability of getting exactly 2 heads equals $\frac{2}{9}$.		
Recognizing the binomial distribution and its parameters: 1 mark Calculating the required probability: 1 mark		
c) The coin is thrown 60 times. **Calculate** the expected value for the number of heads.		1 mark
Y = the number of heads. Y is binomially distributed with parameters $n = 60$ and $p = \frac{1}{3}$. $E(X) = n \cdot p = 60 \cdot \frac{1}{3} = 20$. The expected value for the number of heads is 20.		

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10) A machine produces steel balls. The diameter of the balls is normally distributed with mean $\mu = 18.0$ mm and standard deviation $\sigma = 0.5$ mm. A ball is selected at random. a) **Determine** the probability that its diameter is between 17.0 mm and 19.0 mm.		1 mark
X = the diameter of a ball. a) $P(17.0 \le X \le 19.0) = P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95.$ The probability that the diameter of the ball is between 17.0 mm and 19.0 mm equals about 0.95 or 95 %.		
b) **Determine** the probability that its diameter is between 17.0 mm and 18.5 mm.		2 marks
$P(17.0 \le X \le 18.5) = P(\mu - 2\sigma \le X \le \mu + \sigma) =$ $P(\mu - 2\sigma \le X \le \mu + 2\sigma) - P(\mu + \sigma \le X \le \mu + 2\sigma) \approx$ $0.95 - (0.16 - 0.025) = 0.815.$ Or: $P(17.0 \le X \le 18.5) = P(\mu - 2\sigma \le X \le \mu + \sigma) =$ $P(\mu - \sigma \le X \le \mu + \sigma) + P(\mu - 2\sigma \le X \le \mu - \sigma) \approx$ $0.68 + (0.16 - 0.025) = 0.815.$ Or: $P(17.0 \le X \le 18.5) = P(\mu - 2\sigma \le X \le \mu + \sigma) =$ $P(\mu - 2\sigma \le X \le \mu) + P(\mu \le X \le \mu + \sigma) \approx$ $\frac{0.95}{2} + \frac{0.68}{2} = 0.475 + 0.34 = 0.815.$ The probability that the diameter of the ball is between 17.0 mm and 18.5 mm equals about 0.815 or 81.5 %.		
Developing numeracy: 1 mark Calculating the required probability: 1 mark		

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c) A batch of 400 steel balls is selected at random from this production and the diameter of each ball is measured. If the diameter of a ball is less than 17.0 mm, it will be rejected. **Estimate** how many balls will be rejected.		2 marks
$P(X < 17.0) \approx 0.025$. Let Y be the number of rejected balls among the 400. Y is binomially distributed with parameters $n = 400$ and $p = 0.025$. The number of rejected balls can be estimated by $E(Y) = n \cdot p = 400 \cdot 0.025 = 10$. The number of rejected balls is estimated to be 10.		
Recognizing the binomial distribution and its parameters: 1 mark Calculating the expected value and concluding: 1 mark		